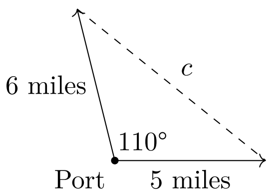
201. After twenty minutes, the situation is as follows:



Using the cosine rule,

$$\begin{aligned} & c^2 = 5^2 + 6^2 - 2 \cdot 5 \cdot 6 \cos 110° \\ \Longrightarrow\ & c = \pm 9.03. \end{aligned}$$

So the distance is 9.03 miles (3sf).

202. The gradient is

$$\frac{\Delta y}{\Delta x} = \frac{2b}{-2b} = -1.$$

By symmetry, such a line is at 45° to the x axis.

203. The possibility space consists of $2^4 = 16$ outcomes. In alphabetical order, successful outcomes are

HHTT	THHT
HTHT	THHT
HTTH	TTHH

This gives $\mathbb{P}(X = 2) = \frac{6}{16} = \frac{3}{8}$.

Nota Bene

The result above is calculated directly by

$$\mathbb{P}(X = 2) = \frac{{}^4\mathrm{C}_2}{2^4} = \frac{3}{8}.$$

204. Factorising,

$$x^2 - 6x + 9 \equiv (x - 3)^2.$$

The presence of a double root means that the curve is tangent to the x axis at $x = 3$.

205. This is not a well-defined function. The reciprocal of 0 is undefined, and 0 is in the proposed domain $\mathbb{R}$, which is the set of all real numbers.

206. The equation of motion is

$$\begin{aligned} & mg - 294 = \tfrac{1}{2}mg \\ \Longrightarrow\ & \tfrac{1}{2}mg = 294 \\ \Longrightarrow\ & m = \frac{2 \cdot 294}{9.8} = 60. \end{aligned}$$

207. The numerator is a difference of two squares:

$$\begin{aligned} & \frac{x^4 - 1}{x^2 - 1} \\ \equiv\ & \frac{(x^2 + 1)(x^2 - 1)}{x^2 - 1} \\ \equiv\ & x^2 + 1. \end{aligned}$$

208. Simplifying the LHS:

$$\begin{aligned} & \mathrm{S}(n - 1) + n \\ \equiv\ & \tfrac{1}{2}(n - 1)n + n \\ \equiv\ & n\left(\tfrac{1}{2}(n - 1) + 1\right) \\ \equiv\ & n\left(\tfrac{1}{2}n + \tfrac{1}{2}\right) \\ \equiv\ & \tfrac{1}{2}n(n + 1) \\ \equiv\ & \mathrm{S}(n). \end{aligned}$$

Nota Bene

This verifies the result, since

$$\begin{aligned} \sum_{r=1}^{n} r &\equiv 1 + 2 + 3 + ... + n - 1 + n \\ &\equiv \left(\sum_{r=1}^{n-1} r\right) + n. \end{aligned}$$

209. ① Nothing special here.

② In this case, the events are complementary, meaning that they are mutually exclusive and exhaustive.

③ Impossible. Consider the formula

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

Since $\mathbb{P}(A \cap B) = 0$ and $\mathbb{P}(A) + \mathbb{P}(B) = \frac{5}{4}$, we have $\mathbb{P}(A \cup B) = \frac{5}{4}$, which is impossible.

210. The exterior angle of a regular n-gon is

$$\theta = \frac{2\pi}{n}.$$

So, in a decagon, $\theta = \frac{2\pi}{10} = \frac{1}{5}\pi$ radians.

211. The mean is $\bar{x} = 1.23$. The standard deviation is $s = 0.831$ (3sf).

Nota Bene

If you got $s = 0.845$ for standard deviation, your calculation used a divisor of $n - 1$ rather than n. This is an *unbiased estimator* of population s.d.

212. (a) Integrating the polynomial, using $+d$ as the constant of integration,

$$\begin{aligned} & \frac{dy}{dx} = 3x^2 + 1 \\ \Longrightarrow\ & y = \int 3x^2 + 1\, dx \\ & \quad = x^3 + x + d. \end{aligned}$$

So, $a = 1$, $b = 0$ and $c = 1$.

(b) Substituting $(-1, 5)$ gives $d = 7$.

213. The ordinal formula for an arithmetic progression (AP) is

$$u_n = a + (n-1)d.$$

This formula also gives

$$u_{n-1} = a + (n-2)d.$$

Substituting these into w_n,

$$\begin{aligned} w_n &= a + (n-2)d + a + (n-1)d \\ &\equiv 2a + 2nd - 3d \\ &\equiv (2a - d) + (n-1)(2d) \end{aligned}$$

This is the ordinal formula for an AP with first term $(2a - d)$ and common difference $2d$.

— Alternative Method —

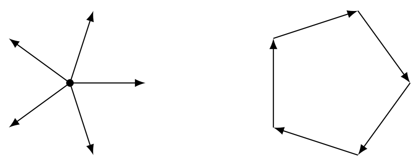
Since u_{n-1} is itself an arithmetic progression, both terms in the ordinal definition of w_n increase by d as n increases by 1. Hence, w_n has a common difference of $2d$, and is an AP.

214. Reflection in $y = x$ changes all the x's for y's and vice versa. So the new line is

$$x = 1 - t, \quad y = 2t, \quad t \in [0, 5].$$

215. If the five forces act along the directions of the sides of a regular pentagon, then the resultant force will be zero:

FORCE DIAGRAM PENTAGON OF FORCES



216. A gradient triangle gives $m = 1$. Substituting this into $y - y_1 = m(x - x_1)$, which is the equation a generic straight line through (x_1, y_1) with gradient m, we get $y = x + 1$.

217. Multiplying out, we have

$$x^3 - x \equiv Ax^3 - Ax^2 + Bx^2 - B.$$

The RHS has no term in x, whereas the LHS does. Hence, no values A, B will make this an identity.

218. (a) The element $a = 2$ must be removed from the domain, to avoid division by zero.

(b) Let $y = \mathrm{f}(x)$. This gives

$$\begin{aligned} y &= \frac{2}{x-4} \\ \Longrightarrow xy - 4y &= 2 \\ \Longrightarrow xy &= 2 + 4y \\ \Longrightarrow x &= \frac{2}{y} + 4. \end{aligned}$$

The inverse has instruction $\mathrm{f}^{-1}(x) = \dfrac{2}{x} + 4$.

219. Subbing the linear equation into the quadratic,

$$\begin{aligned} & x^2 + \left(\tfrac{1}{2}x + 1\right)^2 = 1 \\ \Longrightarrow\ & \tfrac{5}{4}x^2 + x = 0 \\ \Longrightarrow\ & x\left(\tfrac{5}{4}x + 1\right) = 0 \end{aligned}$$

So, (x, y) is $(0, 1)$ or $(-4/5, 3/5)$.

220. The area of the square is 1. The area within a cm of vertex A is a quarter circle of radius a. Its area, therefore, is $\frac{1}{4}\pi a^2$. So we require $\frac{1}{4}\pi a^2 = \frac{1}{2}$. Taking the positive value of a, this rearranges to

$$a = \sqrt{\frac{2}{\pi}}.$$

221. Place the 1 somewhere, without loss of generality. Next, consider the placement of the 2. There are four possible positions for the 2, of which two are adjacent to the 1. So the probability is $\frac{1}{2}$.

— Nota Bene —

"Without loss of generality" (wlog) is a very useful expression in proof. It signifies that the author is making a specific choice between various options, while making it clear to the reader that such a choice won't affect the result in question.

222. The resultant force is $F = 5 \times 2 = 10$ Newtons. Since the 40 N force is 10 N greater than the sum of the other two, there is only one way in which the resultant force can be 10 N. This is with the 20 N and 10 N forces acting in precisely the opposite direction (antiparallel) to the 40 N force.

Furthermore, the forces act on a particle, which, by definition, is located at a single point. So, all three lines of action must be the same.

223. Solving simultaneously, we find intersections at $(-3, -1)$ and $(1/2, 6)$. By Pythagoras, the distance between these two points is $\sqrt{3.5^2 + 7^2} = 7\sqrt{5}/2$.

224. With a vertex at $(1, -3)$, the parabola must have the form $y = k(x-1)^2 - 3$. Substituting $(-2, 0)$ gives $k = \frac{1}{3}$. Multiplying out, the equation of the parabola is $y = \frac{1}{3}x^2 - \frac{2}{3}x - \frac{8}{3}$, which we can write as $3y = x^2 - 2x - 8$.

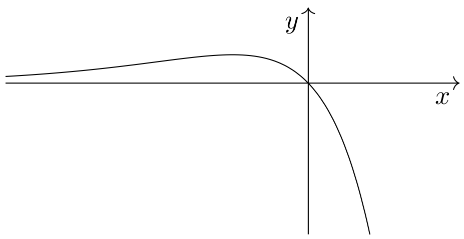
225. We require $r - 1$ non-sixes in the first $r - 1$ rolls, followed by a six. Hence, the probability is

$$\begin{aligned} \mathbb{P}(r) &= \left(\frac{5}{6}\right)^{r-1} \times \frac{1}{6} \\ &\equiv \frac{5^{r-1}}{6^r}, \text{ as required.} \end{aligned}$$

226. (a) As $x \to -\infty$, both exponential terms tend to zero, hence $y \to 0$ and the curve is asymptotic to the x axis.

(b) For x axis intercepts, $2^x - 4^x = 0$. Factorising gives $2^x(1 - 2^x) = 0$, which, since 2^x is strictly positive, is only satisfied by $1 - 2^x = 0$. This gives $x = 0$. So, O is the only axis intercept.

(c) As $x \to \infty$, the curve diverges with $y \to -\infty$. Putting this together with parts (a) and (b), the sketch is



227. Marking outcomes in the possibility space:

	1	2	3	4	5	6
1						
2		✓	✓		✓	
3						
4		✓	✓		✓	
5		✓	✓		✓	
6						

The probability is $\frac{9}{36} = \frac{1}{4}$.

228. The fact that the sequence is an AP is not relevant. The sum of the terms of a sequence is n times its mean. Hence, $n = \frac{420}{10} = 42$.

229. Multiplying top and bottom by -1, we have

$$\frac{x+a}{x-b} \times \frac{x-a}{x+b} \times \frac{x^2-b^2}{x^2-a^2}.$$

The right-hand fraction contains differences of two squares. Each of the factors cancels elsewhere, leaving the expression with constant value 1.

230. Since the graph shown is a derivative, we integrate:

$$\begin{aligned} &\mathrm{f}'(x) = \tfrac{1}{2}x - 1 \\ \Longrightarrow\ &\mathrm{f}(x) = \tfrac{1}{4}x^2 - x + c. \end{aligned}$$

This is a family of parabolae.

231. (a) A rhombus or a square.

(b) 11 is technically a multiple of 11.

232. The boundary L is the perpendicular bisector of the points. The midpoint is $(2, 0)$, so the equation of L is $x = 2$.

233. Rearranging and factorising,

$$\begin{aligned} &x^3 - 4x = 0 \\ \Longrightarrow\ &x(x-2)(x+2) = 0 \\ \Longrightarrow\ &x = 0, \pm 2. \end{aligned}$$

Nota Bene

If your solution was $x = \pm 2$, then you erroneously divided through by x at the beginning. Dividing through by a common factor of x destroys the root $x = 0$. When such a common factor exists, take it out as a factor rather than dividing through by it.

234. The gradients of the lines are $-\frac{2}{3}$ and $\frac{k}{4}$. The lines are perpendicular, so the product of the gradients must be -1. This gives $k = 6$.

235. Using $A_\triangle = \frac{1}{2}ab\sin C$, an equilateral triangle of side length a has area

$$\begin{aligned} A_\triangle &= \tfrac{1}{2}a^2 \sin 60° \\ &\equiv \tfrac{\sqrt{3}}{4}a^2. \end{aligned}$$

So, the surface area of a regular octahedron is

$$\begin{aligned} A_{\text{oct}} &= 8 \times \tfrac{\sqrt{3}}{4}a^2 \\ &\equiv 2\sqrt{3}a^2, \text{ as required.} \end{aligned}$$

236. The standard deviation is irrelevant. In order not to affect the mean, the new datum must have the exact value of the old mean, as any other value would have caused a change in $\bar{x}$. So, $x_{25} = 4$.

237. In this sum, the four input values of the cosine function are symmetrically placed around the unit circle, at

$$\{(90-15)°, (180-15)°, (270-15)°, (360-15)°\}.$$

Hence, their cosines must sum to zero.

Alternative Method

By the symmetry of the cosine graph, the following is an identity:

$$\cos(x + 180°) \equiv -\cos x.$$

The first and third terms of the sum are therefore negatives of each other, as are the second and four terms. Hence, the sum is zero.

238. The roots are $x = \pm a$. The latter factor has no real roots, as $x^2 + b^2$ is strictly positive for all x.

239. From the horizontal information, he will spend half a second crossing the ravine. Assuming a minimal jump, then, he will land after half a second. So, using $s = ut + \frac{1}{2}at^2$,

$$\begin{aligned} &0 = \tfrac{1}{2}u - \tfrac{1}{2}g \cdot \tfrac{1}{4} \\ \Longrightarrow\ &u = \tfrac{1}{4}g = 2.45. \end{aligned}$$

So, he must jump with $u \geq 2.45$ ms^{-1}.

240. The problem is the second line to the third line. The operation "division by x" does not produce a consistent effect on an inequality. If $x = -1$, then $x^2 - 4x > 0$, but $x - 4 \not> 0$.

In a correct solution, we find the roots of the boundary equation $x^2 - 4x = 0$, which are $x = 0, 4$. The solution set is then $x \in (-\infty, 0) \cup (4, \infty)$.

241. Splitting the fraction up,

$$\mathrm{f}'(x) = x^{-\frac{1}{2}} + x^{\frac{1}{2}}.$$

Differentiating again to get the second derivative,

$$\mathrm{f}''(x) = -\tfrac{1}{2}x^{-\frac{3}{2}} + \tfrac{1}{2}x^{-\frac{1}{2}}.$$

Putting this over a common denominator of $2x^{\frac{3}{2}}$,

$$\mathrm{f}''(x) = \frac{x-1}{2x^{\frac{3}{2}}}.$$

242. Since the forces are perpendicular, they combine by Pythagoras. This gives the resultant force as 17 Newtons, so the acceleration is 0.5 ms^{-2}.

Nota Bene

We use "acceleration" to refer to both the **vector** acceleration and its *scalar* magnitude. Compare to **velocity**/*speed* and **displacement**/*distance*, for which there are separate terms for the **vector** and the *scalar*. If there is any room for doubt (there isn't in this question, because we don't know any of the directions involved, so couldn't calculate a vector), we refer to "acceleration" and "magnitude of acceleration".

243. Since $p, q \in \mathbb{Q}$, we know that

$$p = \frac{a}{b}, \quad q = \frac{c}{d},$$

where $a, b, c, d \in \mathbb{Z}$ and b and d are non-zero. So

$$pq = \frac{ac}{bd},$$

which is a quotient (fraction) of integers ac and bd, where bd is non-zero. Hence, $pq \in \mathbb{Q}$. $\square$

244. (a) Yes, because they are not parallel.
(b) No, because they are parallel and distinct.

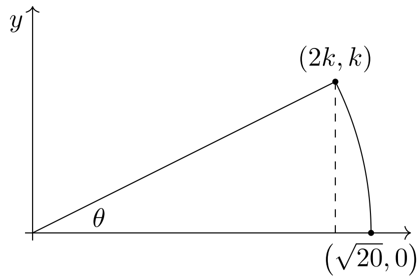
245. Expanding and simplifying,

$$\begin{aligned} &(x^2+1)^2 - (x^2-1)^2 = 0 \\ \Longrightarrow\ & x^4 + 2x^2 + 1 - (x^4 - 2x^2 + 1) = 0 \\ \Longrightarrow\ & 4x^2 = 0 \\ \Longrightarrow\ & x = 0. \end{aligned}$$

246. The speed is given by the integrand; it is 8 ms^{-1}. The duration is $5 - 2 = 3$ seconds. Using $s = ut$, the displacement is $s = 24$ m.

247. If $x + y > 5$, then $2x + 2y > 10$. Adding another y to the LHS can only increase this, since we are told that y is a positive number. Hence, $2x + 3y > 10$.

248. (a) The original point $\big(\sqrt{20}, 0\big)$ is on the x axis and rotation is by $\arctan {}^1\!/\!_2$. So, the gradient of the new radius must be $\tan\theta = \frac{1}{2}$. Hence, the new point must have coordinates of the form $(2k, k)$, for positive k.



(b) Under rotation, the squared distance to the origin remains constant at 20. Pythagoras gives $4k^2 + k^2 = 20$. Solving this and taking the positive value, we get $k = 2$. So, the new coordinates are $(4, 2)$.

249. The annual interest is 4.3%. This is compounded every quarter, at $4.3\% \div 4 = 1.075\%$ interest. This is a scale factor of 1.01075. So, after 6 years, the total amount is $1500 \times 1.01075^{24} = 1938.84$ (2dp). Total interest is £1938.84 − £1500 = £438.84.

250. We are given the derivative $\mathrm{g}'(x)$. The required subtraction is therefore a definite integral:

$$\begin{aligned} \mathrm{g}(6) - \mathrm{g}(0) &= \int_0^6 \mathrm{g}'(x)\,dx \\ &= \int_0^6 \tfrac{1}{3}\,dx \\ &= \Big[\tfrac{1}{3}x\Big]_0^6 \\ &= 2. \end{aligned}$$

251. Since one in every seven integers is a multiple of seven, a run of seven consecutive integers contains a factor of seven. Hence, the product of those seven consecutive integers does too. $\square$

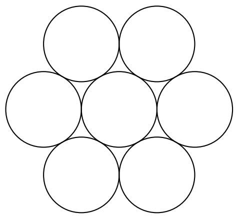
252. Let the hexagon have side length 1. The unshaded triangles have angles $(30°, 60°, 90°)$, and thus sides $({}^1\!/\!_2, {}^{\sqrt{3}}\!/\!_2, 1)$. The rectangle measures ${}^1\!/\!_2 + 1 + {}^1\!/\!_2$ wide by $2 \cdot {}^{\sqrt{3}}\!/\!_2$ high, which is a ratio of $2 : \sqrt{3}$.

253. Rewriting the logarithmic statement as an index statement, we have $y = 3^x$. Then we can write

$$9^x \equiv \left(3^2\right)^x \equiv 3^{2x} \equiv \left(3^x\right)^2 = y^2.$$

254. Pick the letters one at a time. Both must be "not E", so the probability is $\frac{3}{5} \times \frac{2}{4} = \frac{3}{10}$.

255. When three unit circles are placed tangent to each other, their centres form an equilateral triangle, so the angle subtended at each centre is 60°. Since $360° = 6 \times 60°$, precisely six circles will fit around a seventh:



256. (a) Using the polynomial differentiation formula,

$$\frac{dy}{dx} = 4x - 3.$$

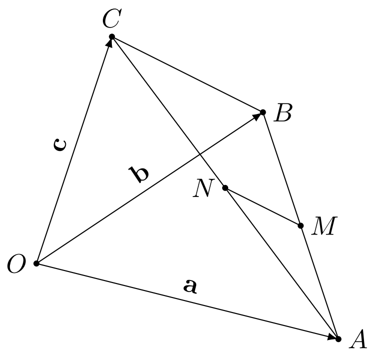
(b) $4x - 3\Big|_{x=2} = 5,$

(c) $2x^2 - 3x + 6\Big|_{x=2} = 8,$

(d) Using $y - y_1 = m(x - x_1)$, the tangent is

$$\begin{aligned} y - 8 &= 5(x-2) \\ \Longrightarrow y &= 5x - 2. \end{aligned}$$

257. Consider position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ of points A, B, C, relative to an arbitrary origin O. The midpoint M of AB is at $\frac{1}{2}(\mathbf{a}+\mathbf{b})$ and the midpoint N of AC is at $\frac{1}{2}(\mathbf{a}+\mathbf{c})$.



The vectors in question are $\overrightarrow{BC} = \mathbf{c} - \mathbf{b}$, and

$$\begin{aligned} \overrightarrow{MN} &= \tfrac{1}{2}(\mathbf{a}+\mathbf{c}) - \tfrac{1}{2}(\mathbf{a}+\mathbf{b}) \\ &\equiv \tfrac{1}{2}(\mathbf{c}-\mathbf{b}) \\ &\equiv \tfrac{1}{2}\overrightarrow{BC}. \end{aligned}$$

Since they are scalar multiples of one another, the vectors $\overrightarrow{MN}$ and $\overrightarrow{BC}$ are parallel. □

258. (a) From the definition,

$${}^n\mathrm{C}_1 \equiv \frac{n!}{1!(n-1)!} \equiv \frac{n(n-1)(n-2)...}{(n-1)(n-2)...}.$$

All of the factors in the denominator cancel, leaving n in the numerator. So, ${}^n\mathrm{C}_1 \equiv n$.

(b) From the definition,

$${}^n\mathrm{C}_2 \equiv \frac{n!}{2!(n-2)!} \equiv \frac{n(n-1)(n-2)(n-3)...}{2(n-2)(n-3)...}.$$

This time, factors of n and $(n-1)$ remain in the numerator, as does 2 in the denominator. So, ${}^n\mathrm{C}_2 \equiv \frac{1}{2}n(n-1)$.

259. The latter statement is equivalent to $x = a, b, c$, written in set notation. So, the statements are those of the factor theorem with three factors. The implication in the factor theorem goes both ways, so the statements should be linked by $\Longleftrightarrow$.

260. We begin with $(x-1)^2$, which must be present to generate the x^2 term. This gives $x^2 - 2x + 1$. Then we add $4(x-1)$ to sort the term in x, giving x^2+2x-3. Lastly, we add 8 to produce the correct constant term:

$$x^2 + 2x + 5 \equiv (x-1)^2 + 4(x-1) + 8.$$

261. The points gives are symmetrical under a switch of x and y coordinates, which is a reflection in the line $y = x$. So, the locus (perpendicular bisector) is the line $y = x$.

— Nota Bene —

The "locus" is simply a technical term for the "set of possible locations". It is Latin for *place.*

262. (a) All three triangles in the diagram are similar. The dashed line is the opposite side in a right-angled triangle with hypotenuse $\cos\theta$, so it has length $\sin\theta\cos\theta$.

(b) Using similar triangles, the lengths are $\sin^2\theta$ and $\cos^2\theta$. Since these add to 1, this verifies the first Pythagorean trig identity:

$$\sin^2\theta + \cos^2\theta \equiv 1.$$

263. For a fraction to be zero, its numerator must be zero. The factor (x^2+q^2) has no real roots, as $q^2 > 0$. This gives roots $x = \pm p$.

— Nota Bene —

If you weren't given the info re the constants being distinct and positive, then the solution would be unclear: if p happened to be equal to $\pm r$, then the roots $x = \pm p$ would appear in the denominator as well as the numerator, rendering the equation undefined at $x = \pm p$.

264. Objects in zero-gravity can exert reaction forces on each other, independently of gravity. When two asteroids collide, each will exert, in response (reaction) to their attempt to pass through each other, a reaction force on the other.

— Nota Bene —

The meanings of the words have changed in the centuries since Newton formulated his system. In particular, his expression of NIII, while obviously correct when he proposed it, is no longer the most comprehensible way of expressing NIII. His famous formulation was:

Every action has an equal and opposite reaction.

But, in modern parlance, a "reaction force" is not simply a force that exists in response to something else, but has a more specific meaning: it means a contact force that is perpendicular to the surfaces in contact. (This is as opposed to friction, which is a contact force parallel to the surfaces in contact.) So, it is now better to remove the word "reaction" from the law altogether. I phrase NIII as follows, symmetrically. It is not so snappy, but better for developing understanding:

Every interaction is modelled with a pair of equal and opposite forces.

265. Since the curve has an asymptote at $x = 4$, its denominator must have a root at $x = 4$. Hence, $16 + 4 + k = 0$, which gives $k = -20$.

266. (a) To 3sf, the percentage error is

$$\frac{\cos\frac{\pi}{24} - \left(1 + \frac{1}{2}\left(\frac{\pi}{24}\right)^2\right)}{\cos\frac{\pi}{24}} = 0.00123\%.$$

(b) Again to 3sf, the percentage error is

$$\frac{\cos\frac{\pi}{6} - \left(1 + \frac{1}{2}\left(\frac{\pi}{6}\right)^2\right)}{\cos\frac{\pi}{6}} = 0.358\%.$$

267. Applying the iteration,

$$\begin{aligned} u_1 &= 2 \cdot 1 + 0 = 2, \\ u_2 &= 2 \cdot 2 + 1 = 5, \\ u_3 &= 2 \cdot 5 + 2 = 12. \end{aligned}$$

268. If a fraction is in its lowest terms, then its roots are the roots of its numerator. Whatever the value of a, the denominator here can have no factors, because $1 + a^2$ must always be positive. So, the roots are exactly the roots of the numerator, which doesn't depend on a.

— Nota Bene —

If a fraction isn't in its lowest terms, then a root in the numerator may not guarantee a root for the fraction. For example:

$$\frac{(x-1)(x-2)}{(x-1)(x-3)} = 0.$$

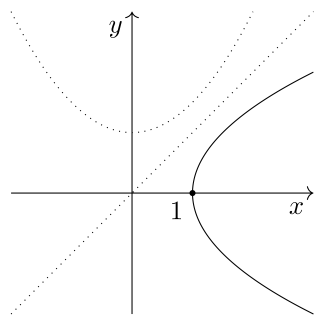
The only root of this equation is $x = 2$, because the fraction is undefined at $x = 1$.

269. The equation of motion is $a = \frac{F}{m}$. The mass m cancels, leaving

$$a = 2t - 2t^2 = 2t(1-t).$$

This is a negative quadratic, so its maximum value is at the vertex. This is midway between the roots, which are $t = 0, 1$. Substituting in $t = \frac{1}{2}$ gives $a_{\text{max}} = 2 \cdot \frac{1}{2} - 2 \cdot \frac{1}{4} = 0.5 \text{ ms}^{-2}$.

270. This is a parabola with vertex at $(1, 0)$. It is a reflection in the line $y = x$ of $y = x^2 + 1$:



271. This is an input (x) transformation. Replacing x with $3x$ stretches the graph by scale factor $1/3$ (i.e. compresses it by scale factor 3) in the x direction. There is no change in the y direction, so the area scale factor is the length scale factor: $\frac{1}{3}$.

272. The interior of a heptagon can be divided up into five triangles. Each of these has interior sum π radians. So, the total is 5π radians.

273. Simplifying the individual surds,

$$\begin{aligned} &\frac{\sqrt{2} - \sqrt{8}}{\sqrt{8} - \sqrt{32}} \\ &= \frac{\sqrt{2} - 2\sqrt{2}}{2\sqrt{2} - 4\sqrt{2}} \\ &= \frac{-\sqrt{2}}{-2\sqrt{2}} \\ &= \frac{1}{2}. \end{aligned}$$

274. 3D Pythagoras is the same as 2D Pythagoras, just with more terms. So,

$$d = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}.$$

275. (a) Values in $[0, 1]$ square to give values in $[0, 1]$. So, the range is $[0, 1]$.

(b) Squares are non-negative, so the range is the same as in part (a), i.e. $[0, 1]$.

(c) With logic as in (b), the range is $[0, \infty)$.

276. The perpendicular bisector passes through the midpoint (mean) of the two points, which is $(6, 2)$. Its gradient is the negative reciprocal of $-\frac{1}{3}$, which is 3. So, its equation is $y = 3x - 16$.

277. The difference between the first term and last term is $l - a$, and there are $n - 1$ steps between the first and the nth term. Since the sequence is an AP, each of these steps is a common difference, giving the required result.

278. Putting the fraction in its lowest terms,

$$\frac{x+b}{x+b-1} = 0.$$

The fraction is zero iff its numerator is zero. So, the solution is $x = -b$.

— Alternative Method —

The numerator is zero at $x = a, -b$. However, the denominator is also zero at $x = a$, at which value the LHS is undefined. Hence, the only root of the equation is $x = -b$.

279. We integrate twice, indefinitely. If $\mathrm{g}''(x) = 2$, then $\mathrm{g}'(x) = 2x + c$, and $\mathrm{g}(x) = x^2 + cx + d$, for some constants c and d. Since $\mathrm{g}(x)$ is a quadratic with leading coefficient 1, the equation $y = \mathrm{f}(x)$ defines a monic parabola.

280. The possibility space, with the first score on the horizontal axis and successful outcomes ticked, is:

	1	2	3	4	5	6
1						
2	✓					
3	✓	✓				
4	✓	✓	✓			
5	✓	✓	✓	✓		
6	✓	✓	✓	✓	✓	

The probability is $p = \frac{15}{36} = \frac{5}{12}$.

281. NII horizontally is $T\cos\theta - 10 = 0$, which is $T\cos\theta = 10$. NII vertically is $T\sin\theta = 5 \times 2 = 10$. Dividing the latter by the former,

$$\begin{aligned} \frac{T\sin\theta}{T\cos\theta} &= \frac{10}{10} \\ \implies \tan\theta &= 1. \end{aligned}$$

So, $\theta = 45°$ and $T = 10\sqrt{2}$.

— Nota Bene —

The identity used above is one of the two key identities in trigonometry. The other is the first Pythagorean trig identity:

$$\begin{aligned} \tan\theta &\equiv \frac{\sin\theta}{\cos\theta}, \\ \sin^2\theta + \cos^2\theta &\equiv 1. \end{aligned}$$

282. An ellipse is a closed curve, so evaluating the LHS and comparing to 100 will determine whether a point is inside, on, or outside it:

$$2x^2 + 3y^2 \Big|_{(4,5)} = 107 > 100.$$

So, the point lies outside the ellipse.

283. (a) The mean of the side lengths is $\frac{35}{5} = 7$. Since the pentagon is irregular, the longest must be longer than this, giving $l > 7$.

(b) Since the sides are in AP, however much greater than the mean the largest is, the smallest must be that much smaller than the mean. In other words, the shortest and longest values must be equidistant from the mean. The smallest must be greater than 0, and the mean is 7, so the largest must be less than 14. This gives $l < 14$.

284. $\sum x$ for the twenty pupils is $69.3 \times 20 = 1386$. Adding 86 gives a new sum of $1386 + 86 = 1472$. The new mean is

$$\frac{1472}{21} = 70.1\% \text{ (1dp)}.$$

285. Because the numerator is never zero, this curve has vertical asymptotes wherever the denominator is zero. Setting $x^3 - px = 0$ gives $x = 0, \pm\sqrt{p}$, so these are the equations of the asymptotes.

286. (a) False, because $x^3 = y^3 \not\Leftarrow x = -y$.

(b) True.

(c) False, because $x^5 = y^5 \not\Leftarrow x = -y$.

287. The region is circular, with radius $\sqrt{5}$. The centre isn't relevant to the area, which is 5π.

288. The derivative of the first curve is $\frac{dy}{dx} = 3x^2 + 2x$. Multiplying out the second curve,

$$\begin{aligned} y &= (\sqrt{x}+1)(\sqrt{x}-1) \\ &= x - 1. \end{aligned}$$

This gives $\frac{dy}{dx} = 1$. Equating the derivatives,

$$\begin{aligned} & 3x^2 + 2x = 1 \\ \Longrightarrow\ & 3x^2 + 2x - 1 = 0 \\ \Longrightarrow\ & (3x-1)(x+1) = 0 \\ \Longrightarrow\ & x = \tfrac{1}{3}, -1. \end{aligned}$$

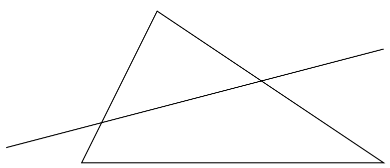
289. Assuming that the domain is $\mathbb{R}$, the range of $\sin x$ is $[-1,1]$. So, the range of $(\sin x + 1)$ is $[0,2]$. Hence, the range of $(\sin x + 1)^2$ is $[0,4]$, and the range of $(\sin x + 1)^2 + 1$ is $[1,5]$.

290. (a) The perpendicular bisectors are
   i. $y = -\frac{1}{2}x + \frac{5}{2}$
   ii. $x = 1$.
   (b) Subbing $x = 1$, point X is at $(1,2)$.
   (c) By Pythagoras, each of A, B, C, D lies at a squared distance of $2^2 + 1^2 = 5$ from X. So, as required, $|AX| = |BX| = |CX| = |DX|$.
   (d) From part (c), X is the centre of a circle of radius $\sqrt{5}$ passing through A, B, C and D. $ABCD$, therefore, is a cyclic quadrilateral.

291. Any two functions differing by a constant provide a counterexample, e.g. $\mathrm{f}(x) = x^2$ and $\mathrm{g}(x) = x^2 + 1$.

292. The intersection of the two straight lines is $(-1,1)$. Substituting this into the LHS of the circle gives $(-1)^2 + 1^2 = 2$, so this intersection also lies on the circle. Hence, the three graphs are concurrent.

293. Consider side S of one of the triangles. Since the total number $n \in \mathbb{N}$ of intersections is finite, S intersects the other triangle finitely many times. Assume that S intersects the other triangle twice. Then it cannot intersect a third time, because a triangle is convex: once outside the triangle, a line cannot re-enter it.



Each side can produce at most two intersections, so there can be at most six intersections overall. Hence, $n \leq 6$. □

294. Rearranging the second equation,

$$t = \frac{v-u}{a}.$$

We multiply the first equation by t and substitute the second in. This gives

$$\tfrac{1}{2}(u+v) \cdot \frac{v-u}{a} = s.$$

Multiplying by $2a$ and expanding the difference of two squares gives $v^2 - u^2 = 2as$. The required result $v^2 = u^2 + 2as$ follows directly. □

— Nota Bene —

This formula, being the only *suvat* equation not to mention time, is an *energy equation*. Multiplying it by $\frac{1}{2}m$ gives the *work-energy* principle, which links the change in *kinetic energy* to the *work done*:

$$\begin{aligned} & \tfrac{1}{2}mv^2 = \tfrac{1}{2}mu^2 + mas \\ \Longleftrightarrow\ & \underbrace{\tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2}_{\Delta\text{KE}} = \underbrace{Fs}_{\text{Work}}. \end{aligned}$$

295. The equation has one root at $x = 0$. We require the remaining quadratic factor $(x^2 + kx + 9)$ to have exactly one root. So, we set $\Delta = k^2 - 36 = 0$, giving $k = \pm 6$.

296. The triangles are congruent (condition SAS), as they share the same angles and a length. Hence $a = c$ and $b = d$. So,

$$m_1 m_2 = \frac{-a}{b} \times \frac{d}{c} = \frac{-a}{b} \times \frac{b}{a} = -1.$$

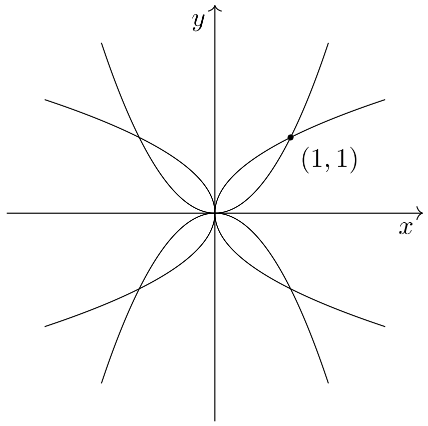
297. (a) $\displaystyle\int 0\,dt = c$,
   (b) $\displaystyle\int 1\,dt = t + c$,
   (c) $\displaystyle\int \sqrt{t}\,dt = \tfrac{2}{3}t^{\frac{3}{2}} + c$.

298. Consider the extreme/boundary cases:
   - In an isosceles triangle (a, a, b) with two sides of length a just shorter than 6, the shortest side b can be made as small as is required. It cannot have zero length, however, or the triangle ceases to be a triangle.
   - The shortest side is as long as possible when all the sides are the same length. This makes an equilateral triangle of side length 4 units. This boundary is attainable.

All values between the bounds are attainable. So, in interval set notation, the set of possible values for the length of the shortest side is $(0, 4]$.

299. (a) False: a square is also a kite.
   (b) True.
   (c) True: a square is also a trapezium.

300. The curve $y = x^2$ is the standard parabola. The other three curves are reflections of this in the lines $y = 0$ and $y = \pm x$:



—— End of 3rd Hundred ——